Name——————————, **ID** ——————

Exam I, MTH 205, Fall 2015

Ayman Badawi

QUESTION 1. CLEARLY circle the correct answer:

(i) $\ell\{(x-1)^2\} =$

(a) $\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}$ (b) $\frac{2}{(s-1)^3}$ (c) $\frac{e^s}{s^2}$ (d) Something else

(ii) $\ell\{U(x-3)sin(x-3)\}$

(a) $\frac{e^{-3s}}{s^2+1}$ (b) $\frac{e^{-3s}}{(s+3)^2+1}$ (c) $\frac{e^{-3s}}{(s-3)^2+1}$ (d) something else

(iii) $\ell\{\int_0^x \ sin(r)e^{3r}\ dr\}$:

(a) $\frac{1}{s((s-3)^2+1)}$ (b) $\frac{1}{(s^2+1)(s-3)}$ (c) $\frac{1}{((s-3)^2+1)(s-3)}$ (d) Something else

(iv) $\ell^{-1}\{\frac{s}{s^2+2s+2}\}$

(a) $cos(x)e^{-x}$ (b) $cos(x)e^{-x} - sin(x)e^{-x}$ (c) $cos(x)e^{x}$ (d) $c_1e^{-x} + c_2xe^{-x}$ for some constants c_1, c_2 (e) Something else

(v) Given $y = 3cos(2x)$ is a solution to the diff. equation $y^{(2)} + ay' + by = 0$, where a, b are some constants. Then the values of a, b are

(a) $a = 0$, $b = 3$ (b) $a = 0, b = 4$ (c) $a = 3, b = 4$ (d) $a = 3, b = 2$ (d) there are infinitely many values for a, b, more info. is needed

(vi) Given $2x^2e^{-x}$ is a particular solution to the diff. equation $y^{(2)} + ay' + by = 4e^{-x}$ for some constant a, b. Then the values of a, b are

(a) $a = -1, b = 2$ (b) $a = 2, b = 1$ (c) $a = 4, b = 1$ (d) $a = 2, b = -1$ (e) Something else

(vii) A particular solution to the diff. equation $y' + 2y = 1 - \int_0^x y(r)\ dr$ is

(a) $y_p = x^2e^{-x}$ (b) $y_p = e^{-x}$ (c) $y_p = xe^{-x}$ (d) $y_p = 2x$ (e) Something else

(viii) The solution to the diff. equation $y^{(2)} - 4y' + 4y = U(x-3)e^{(2x-6)}, y(0) = y'(0) = 0$ is

(a) $y = u(x-3)x^2e^{2x}$ (b) $y = 0.5U(x-3)x^2e^{2x}$ (c) $y = 0.5U(x-3)(x-3)^2e^{(2x-6)}$ (d) Something else

(ix) The general solution to the diff. equation $y^{(5)} + y^{(3)} = 0$ is

(a) $y = c_1+c_2x+c_3x^2+c_4cos(x)+c_5sin(x)$ (b) $y = c_1+c_2sin(x)+c_3xsin(x)$ (c) $y = c_1+c_2cos(x)+c_3sin(x)$ (d) Something else

(x) $\ell^{-1}\{\frac{s+2}{s^2-3s+2}\} =$

(a) $4e^{2x} + e^x$ (b) $4e^{2x} - 3e^x$ (c) $4e^x + 3e^{2x}$ (d) $cos(x)e^{-2x}$ (e) Something else

(xi) $\ell^{-1}\{\frac{3s+4}{(s+1)^2}\} =$

(a) $3x^2e^{-x} + 4xe^{-x}$ (b) $3e^{-x} + xe^{-x}$ (c) $3xe^{-x} + 4x^2e^{-x}$ (d) Something else

(xii) Given that $y^{(2)} + y = 0$, has infinitely many solutions when $y(\pi/2) = 1$ and $y'(\pi) = a$, for some constant a. Then the value of a is,

(a) 1 (b) -1 (c) can be any real number (d) Something else

(xiii) The largest interval around x where the diff. equation $\sqrt{x+3}y^{(2)} + \frac{1}{x-6}y' + xy = 3, y(0) = y'(0) = 1$ has a unique solution is

(a) $(-3,\infty)$ (b) $(6,\infty)$ (c) $(-3,6)$ (d) Something else

(xiv) $\ell\{x^2 3^x\} =$

(a) $\frac{2}{(s-3)^3}$ (b) $\frac{2}{(s-3)^2}$ (c) $\frac{2}{(s-\sqrt{3})^3}$ (d) Something else

(xv) The solution to the diff. equation $y' + 2y = e^{-2x} - \int_0^x e^{-2r}y(x-r)\,dr, y(0) = 0$ is

(a) $y = sin(x)e^{-2x}$ (b) $y = x^2e^{-2x}$ (c) $y = xe^{-2x}$ (d) Something else

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Exam II, MTH 205, Fall 2015

Ayman Badawi

Excellent 50/50

QUESTION 1. (i) Find the general solution to the Diff. Equation $(2x+1)y' - y = y^3(2x+1)e^{(-2x^2-2x+7)}$

bernouli

$$y' - \frac{1}{(2x+1)}y = e^{-2x^2-2x+7}y^3$$

$n=3$

$w = y^{1-n} = y^{-2}$

$$\Rightarrow w' - \frac{(1-3)}{(2x+1)}w = (1-3)e^{-2x^2-2x+7}$$

$$\Rightarrow w' + \frac{2}{2x+1}w = -2e^{-2x^2-2x+7}$$

$Q(x) = \frac{2}{2x+1}$, $K(x) = -2e^{-2x^2-2x+7}$

$$\Rightarrow w = \frac{\int K(x)e^{\int Q(x)dx}}{e^{\int Q(x)dx}} = \frac{\int -2e^{-2x^2-2x+7}e^{\int\frac{2}{2x+1}dx}}{e^{\int\frac{2}{2x+1}dx}}$$

$$= \frac{\int -2(2x+1)e^{-2x^2-2x+7}dx}{2x+1}$$

let $u = -2x^2-2x+7$, $du = (-4x-2)$, $= -2(2x+1)$

$$= \frac{e^{-2x^2-2x+7}+C}{2x+1}$$

$$\Rightarrow w = y^{-2} = \frac{e^{-2x^2-2x+7}+C}{2x+1}$$

$$\Rightarrow \boxed{y = \sqrt{\left(\frac{2x+1}{e^{-2x^2-2x+7}+C}\right)}}$$

20/20

(ii) Find the general solution to the Diff. Equation $x^2y^{(2)} + xy' + y = ln(x)$ → $y'' + \frac{y'}{x} + \frac{y}{x^2} = \frac{\ln x}{x^2}$

Cauchy

⇒ for y_h ⇒ let $y = x^n$, $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$

⇒ so $[n(n-1)+n+1]x^n = 0$

⇒ so $n(n-1)+n+1=0$

⇒ $n^2 - n + n + 1 = 0$

⇒ $n^2+1=0$

⇒ $n = \pm i$

⇒ so $y_h = c_1 \cos(\ln x) + c_2 \sin(\ln x)$

$-u\cos u + \sin u$

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⇒ for y_p ⇒ let $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$, $k(x) = \frac{\ln x}{x^2}$

⇒ $w(y_1, y_2) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}$

⇒ $y_p = y_2 \int \frac{y_1 k(x)}{w(y_1,y_2)} dx - y_1 \int \frac{y_2 k(x)}{w(y_1,y_2)} dx$

$= \sin(\ln x) \cdot \int \frac{\cos(\ln x) \ln x / x^2}{1/x} dx - \cos(\ln x) \int \frac{\sin(\ln x) \ln x / x^2}{1/x} dx$

$= \sin(\ln x) \int \frac{\ln x \cos(\ln x)}{x} dx - \cos(\ln x) \int \frac{\ln x \sin(\ln x)}{x} dx$

$= \sin(\ln x) \cdot (\ln(x) \sin(\ln x) + \cos(\ln x)) - \cos(\ln x) \cdot (-\ln(x)\cos(\ln x) + \sin(\ln x))$

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$$= \ln(x)\left(\sin^2(\ln x) + \cos^2(\ln x)\right) + \cos(\ln x)\sin(\ln x) - \cos(\ln x)\sin(\ln x)$$

$$= \ln(x)$$

$$\Rightarrow y_p = \ln x$$

$$\Rightarrow y = y_h + y_p = \boxed{\ln x + c_1 \cos(\ln x) + c_2 \sin(\ln x)}$$

10/10

(iii) Find the solution to the Diff. equation $y' - \frac{1}{x}y = (1 + x\ln(x))e^x$, $y(1) = 4$

$$\Rightarrow \underbrace{y' - \tfrac{1}{x}}_{G(x)} y = \underbrace{(1 + x\ln x)e^x}_{K(x)}$$

$$\Rightarrow y = \frac{\int K(x) e^{\int G(x)dx}}{e^{\int G(x)dx}} = \frac{\int (1+x\ln x)e^x \cdot e^{-\int \frac{1}{x}dx} dx}{e^{-\int \frac{1}{x}dx}}$$

$$= \frac{\int \frac{(1+x\ln x)e^x}{x} dx}{1/x}$$

$$\Rightarrow \int (f(x) + f'(x))e^x dx = f(x)e^x + c$$

$$= \frac{\int (\frac{1}{x} + \ln x)e^x dx}{1/x}$$

$$= \frac{(\ln x)e^x + c}{1/x}$$

$$y = (x\ln x)e^x + Cx$$

$$\Rightarrow 4 = y(1) = (1 \cdot \ln 1)e^1 + C \quad (\ln 1 = 0)$$

$$\Rightarrow C = 4$$

$$\Rightarrow \boxed{y = (x\ln x)e^x + 4x}$$

10/10

$$y_p = y_2 \int \frac{y_1 k(x)}{W(y_1, y_2)}\,dx - y_1 \int \frac{y_2 k(x)}{W(y_1, y_2)}\,dx$$

$$= e^x \cdot \int \frac{(x+1)\cdot(xe^x)}{(xe^x)}\,dx - (x+1) \int \frac{e^x \cdot (xe^x)}{(xe^x)}\,dx$$

$$= e^x \cdot \int (x+1)\,dx - (x+1) \int e^x\,dx$$

$$= e^x \cdot \left(\frac{x^2}{2} + x\right) - (x+1)e^x = e^x \cdot \left(\frac{x^2}{2} - 1\right)$$

$$\Rightarrow y = y_h + y_p = \boxed{c_1(x+1) + c_2 e^x + e^x\left(\frac{x^2}{2} - 1\right)}$$

(iv) Find the general solution to the Diff. Equation $xy^{(2)} - (x+1)y' + y = x^2e^x$, given $y = -e^x$ is a solution to the homogeneous part.

[redacted]

$$y'' - \underbrace{(1 + 1/x)}_{Q(x)} y' + \frac{y}{x} = xe^x$$

$\Rightarrow$ for y_h, let $y_1 = -e^x$

$$\Rightarrow \text{so } y_2 = y_1 \int \frac{e^{-\int Q(x)dx}}{y_1^2} dx$$

$$= -e^x \cdot \int \frac{e^{\int (1 + 1/x)dx}}{e^{2x}} dx$$

$$= -e^x \cdot \int \frac{e^{x + \ln x}}{e^{2x}} dx$$

$$= e^x \cdot \int \frac{xe^x}{e^{2x}} dx$$

$$= -e^x \cdot \int xe^{-x} dx$$

$$= -e^x \cdot \left(-(x+1)e^{-x}\right) = (x+1)$$

$$\Rightarrow \text{so } y_h = c_1(x+1) + c_2 e^x$$

$\Rightarrow$ for y_p, let $y_1 = (x+1)$, $y_2 = e^x$, $k(x) = xe^x$

$$\Rightarrow W(y_1, y_2) = \begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = xe^x + e^x - e^x = xe^x$$

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(v) A 39.2 is attached to a spring having a spring constant 4N/m. At t = 0, the object is released from a point 1.5 meter below the equilibrium position with an upward velocity 1m/s and with constant external force F(t) = 14.

a) Find the equation of the motion, $x(t)$.

$\Rightarrow x'' + \frac{a}{m}x' + \frac{k}{m}x = \frac{F(t)}{m}$ $\Rightarrow m = \frac{39.2}{9.8} = 4\text{kg}$

$\Rightarrow a = 0$

$\Rightarrow x'' + x = \frac{14}{4}$ $\Rightarrow k = 4$

$\Rightarrow x(0) = 1.5\text{m}$

$\Rightarrow x'' + x = \frac{7}{2}$ $\Rightarrow x'(0) = -1\text{m/s}$

$\Rightarrow$ for x_h, let $x = e^{mt}$ so $m^2 + 1 = 0$, $m = \pm i$

$\Rightarrow x_h = c_1 \cos t + c_2 \sin t$

$\Rightarrow$ for x_p, let $x = A$ so $A = \frac{7}{2} = x_p$

$\Rightarrow x(t) = \frac{7}{2} + c_1 \cos t + c_2 \sin t$

$\Rightarrow 1.5 = x(0) = 3.5 + c_1$

$\Rightarrow c_1 = -2$

$\Rightarrow x'(t) = 2\sin t + c_2 \cos t$

$\Rightarrow -1 = x'(0) = c_2$

$\Rightarrow c_2 = -1$

$\Rightarrow x(t) = 3.5 - (2\cos t + \sin t)$

b) Find the phase angle ϕ and rewrite $x(t)$ using the angle ϕ.

$x(t) = 3.5 - \sqrt{2^2 + 1^2} \cdot \cos(t - \tan^{-1}(\frac{1}{2}))$

$= 3.5 - \sqrt{5}\cos(t - 0.46365)$

$\Rightarrow \varphi = \tan^{-1}(\frac{1}{2}) = 0.46365$

c) I claim that the object will stay below the equilibrium point at any time t. Justify my claim or prove me wrong.

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d) If my claim is correct as in (c), then what should the maximum constant external force be so that the object will pass through the equilibrium point?

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for c) and d) flip page ↓

$\Rightarrow$ c) $x(t) = 3.5 - \sqrt{5}\cos(t - 0.46365) \geq 0$

because ↓

$3.5-\sqrt{5}(1) \leq 3.5 - \sqrt{5}\cos(t-0.46365) \leq 3.5-\sqrt{5}(-1)$

$\underbrace{1.264}_{>0} \leq 3.5 - \sqrt{5}\cos(t-0.46365) \leq \underbrace{5.736}_{>0}$

or if $3.5-\sqrt{5}\cos(t-0.46365) = 0$

$\Rightarrow \cos(t-0.46365) = \frac{3.5}{\sqrt{5}} = 1.5652 > 1$

impossible!! so the object never passes through equilibrium

$\Rightarrow$ d)

for the object to pass through equilibrium,

$\frac{F}{4} - \sqrt{5}\cos(t-0.46365) = 0$

$\Rightarrow \cos(t-0.46365) = \frac{F}{4\sqrt{5}} \leq 1$

so $F \leq 4\sqrt{5} = 8.944\ N$

$\Rightarrow F_{max} = 4\sqrt{5} = 8.944N$ ✓

Name————————————, ID ———————

Final Exam , MTH 205, Fall 2015

Ayman Badawi

QUESTION 1. (8 points) Consider the differential equation $(2xy+4)dx+(x^2-y^2)dy = 0$.

a)Check whether the equation is exact.

b)Find the general solution (i.e., $y(x)$) to the equation.

QUESTION 2. (10 points) Find the general solution in explicit form to the differential equation $\dfrac{dy}{dx} = y^2e^{-x}$

QUESTION 3. (8 points) Consider the differential equation $\dfrac{dy}{dx} = y^3 + 2y^2 + y$.

a) Find all equilibrium points of this nonlinear differential equation and classify each as stable, semi-stable or unstable.

b) If $y(0) = -4$, then find $lim_{x\to\infty} y(x)$

QUESTION 4. (8 points) (a) Find the general solution to $y^{(5)} + 2y^{(4)} + y^{(3)} = 0$.

b) For the differential equation $y^{(5)} + 2y^{(4)} + y^{(3)} = 20 + (x^2 + x^3)e^{-x}$ write down the form of y_p but do not find it.

QUESTION 5. (8 points) Solve for y(x): $y' + 2y = 1 - \int_0^x y(r)dr, y(0) = 0.$

QUESTION 6. (10 points) Find the general solution to $x^2y^{(2)} + xy' + y = sec(lnx)$

QUESTION 7. (8 points) A water tank initially contains 300 gallons of pure water. Brine with a concentration of 3 pounds per gallon is being pumped into the tank at a rate of K gallons per minute where $K > 0$ is some constant. The well mixed solution is pumped out at the same rate (i.e., K gallons per minute).

a) Find $A(t)$ (amount of salt at any time t, where t is time in minutes), note that you need to write $A(t)$ in terms of t and K.

b) Given that the amount of salt in the tank after 5 hours is 450 pounds find the value of K.

QUESTION 8. (8 points) An object weighing 8 pounds stretches a spring by 2 feet.

a) Find the mass and the spring constant. (note that gravity = $g = 32ft/sec^2$).

b) Find the equation of motion $x(t)$ if the object is released from the equilibrium position with downward velocity of 1 ft/sec.

c) Rewrite $x(t)$ in terms of the phase angle.

c) Let L be the maximum distance that the object reaches below the equilibrium point and G be the maximum distance that the object reaches above the equilibrium point. Find L and G.

QUESTION 9. (21 points) Find the following transformations:

(i) $\ell\{(x+e^x)^2\}$

(ii) $\ell\{U(x-\pi)sin(x)\}$.

iii) $\ell\{x\delta(x-2)\}$.

iv) $\ell\{f(x)\}$, where $f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ e^{(1-x)} & \text{if} \quad 1 \le x < \infty \end{cases}$.

(v) $\ell^{-1}\{\frac{s+4}{s^2+4s+5}\}$

vi) $\ell^{-1}\{\frac{e^{-\pi s}}{s^2+4}\}$

vii) $\ell^{-1}\{\frac{3s}{(s^2+9)^2}\}$ (Use convolution)

QUESTION 10. (4 points) Find the largest interval for which the initial value problem:

$$\sqrt{x+6}y^{(2)}+\frac{3}{x-10}y'+6y=\frac{1}{x-5}, y(-3)=0, y'(-3)=1$$

has a unique solution.

QUESTION 11. (8 points) Solve for $x(t), y(t)$: $y'-x=0$ and $y+x'=t$, where $y(0)=0$ and $x(0)=1$

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